ON 2-SYLOW INTERSECTIONS

BY

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ABSTRACT

Let G be a finite group. We say that G satisfies $I(k)$ if any $k+1$ distinct Sylow 2-subgroups of G interesect trivially. In this paper we classify all finite groups satisfying $I(3)$, and all finite simple groups satisfying $I(5)$.

THEOREM 1. *Let G be a finite group in which any five distinct Sylow 2-subgroups intersect trivially. Then one of the following holds.*

(1) *$O'(G)/O(G)$, the smallest normal subgroup of $G/O(G)$ of odd index, is isomorphic to one of the following groups:*

(i) *a Sylow 2-subgroup of G;*

(ii) *$GL_2(3)$, $SL_2(3)$, $L_2(3)$, or an extension of rank 1 of such a group;*

(iii) *the extension of a 2-group by $L_2(q)$, $Sz(q)$, $U_3(q)$, q even;*

(iv) *$PGL_2(3)$, $PGL_2(5)$, $L_2(11)$ or $L_2(13)$.*

(2) *G contains three Sylow 2-subgroups, any two of which intersect in $O_2(G)$ and $O_{2,2,2}(G) = G$.*

COROLLARY 2.*Let G be a finite simple group satisfying the assumption of Theorem 1. Then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$, $U_3(q)$ for q even, $L_2(11)$ or $L_2(13)$.*

THEOREM 3. *Let G be a finite simple group in which any seven Sylow 2-subgroups intersect trivially. Then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$, $U_3(q)$, q even, $L_2(7)$, $L_2(9)$, $L_2(11)$, $L_2(13)$, $L_2(19)$, $L_3(3)$, M_{11} or J_1.*

We shall adopt the following conventions. All groups considered are finite,

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$H \leqq G$ ($H < G$) means that H is a (proper) subgroup of G, $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G. By $A \equiv B$ we mean that A is defined by B.

Let P be a p-subgroup of the group G. Define $\Sigma(P) \equiv \{S \in \mathrm{Syl}_p(G) | S \geqq P\}$, $\sigma(P) \equiv |\Sigma(P)|$. Define $\bar{P} \equiv \cap \Sigma(P)$ and $C_p(G) \equiv \{P \text{ a } p\text{-subgroup of } G \,|\, P = \bar{P}\}$. $C_p(G)$ consists of p-Sylow intersections in G. Clearly $O_p(G) \in C_p(G)$, and $\sigma(P) = 1$ for $P \in C_p(G)$ implies $P \in \mathrm{Syl}_p(G)$. For $P \in C_p(G)$, define $\min P \equiv \{P' \in C_p(G) \,|\, P' > P$ and no $X \in C_p(G)$ satisfies $P' > X > P\}$. It is clear that $\min P = \varnothing$ if and only if $P \in \mathrm{Syl}_p(G)$.

Define by $\min_k P \equiv \{P' \in \min P \,|\, \sigma(P') = k\}$. Denote $\min O_p(G)$ by $\min G$. Denote by $I(k)$ the class of all finite groups G with the property that any $k+1$ Sylow 2-subgroups intersect trivially.

Lemma 4. *Let $P, P' \in C_p(G)$. If $P' \geqq P$ then $N_{P'}(P)/P$ acts faithfully on $\Sigma(P)$. If, in addition, $P' \in \min P$ then any $gP \in (N_{P'}(P)/P)^{\#}$ acts fixed-point freely on $\Sigma(P) \setminus \Sigma(P')$.*

Proof. It is clear that $N_{P'}(P)$ acts on $\Sigma(P)$ by conjugation. If $g \in N_{P'}(P)$ stabilizes some $S \in \Sigma(P)$ then g is a p-element of $N_G(S)$, and as such belongs to S. Therefore $P = \bar{P} \equiv \cap\Sigma(P)$ implies that $N_{P'}(P)/P$ acts faithfully on $\Sigma(P)$.

If $P' \in \min P$ and $S \in \Sigma(P)$ is stabilized by $gP \in (N_{P'}(P)/P)$, then $P < \langle P, g\rangle \leqq P' \cap S$ implies $S \in \Sigma(P')$ as $P' \cap S \in C_p(G)$.

Corollary 5. *If $P' \in \min P$ then $|N_{P'}(P)/P|$ divides $\sigma(P) - \sigma(P')$.*

Proof. Consider $N_{P'}(P)/P$ as a faithful permutation group on $\Sigma(P) \setminus \Sigma(P')$. As such any non-identity element acts fixed-point freely, hence $|N_{P'}(P)/P|$ divides $\sigma(P) - \sigma(P')$.

Lemma 6. *If P is a p-subgroup of G then $\sigma(P) \equiv 1 \pmod p$.*

Proof. For every p-subgroup P of G, $\sigma(P) = \sigma(\bar{P})$, so we may assume that $P \in C_p(G)$. We shall prove the lemma by induction on $|G|/|P|$. If $P \in \mathrm{Syl}_p(G)$ then $\sigma(P) = 1 \equiv 1 \pmod p$. If $P \notin \mathrm{Syl}_p(G)$ then $\min P \neq \varnothing$. Let $P' \in \min P$. Clearly $|P'| > |P|$. Thus, by the induction hypothesis, $\sigma(P') \equiv 1 \pmod p$, and hence by Corollary 5 $\sigma(P) \equiv \sigma(P') \pmod p \equiv 1 \pmod p$.

Corollary. 7. *Let $T \in C_2(G)$. If $\sigma(T) = 3$ then* $\min T = \Sigma(T)$.

Proof. Let $T' \in \min T$. As $T' > T$ and $T \in C_2(G)$, it follows that $\Sigma(T') \subset \Sigma(T)$. Hence $3 = \sigma(T) > \sigma(T') \geqq 1$ whence by Lemma 6 $\sigma(T') = 1$, $T' \in \mathrm{Syl}\ (G)$, and $T' \in \Sigma(T)$. Hence $\min T \subseteq \Sigma(T)$.

Now assume that for some $S \in \Sigma(T)$, $S \notin \min T$. In that case there exists $S' \in \Sigma(T)$ with $S > S \cap S' > T$. But then $1 = \sigma(S) < \sigma(S \cap S') < \sigma(T) = 3$ in contradiction to Lemma 6. Hence $\min T \supseteq \Sigma(T)$ and we are through.

LEMMA 8. *Let $O_p(G) = 1$. If $P \in \min G$ then:*

(i) *$P \cap P^g \neq 1$ implies $P = P^g$,*

(ii) *P is normal in $S \in \Sigma(P)$ iff $P \cap Z(S) \neq 1$, and*

(iii) *P is not normal in $S \in \Sigma(P)$ implies P is isomorphic to a subgroup of $N_S(P)/P$.*

PROOF. If $1 < P \cap P^g \leqq P$ it follows, by the definition, that $P \cap P^g = P$, hence $P = P^g$. The second assertion is obvious because for any $z \in P$, $C_G(z) \leqq N_G(P)$ by (i). If P is not normal in $S \in \Sigma(P)$ then there exists $g \in N_S(N_S(P)) \setminus N_S(P)$. Then $P \neq P^g$ which is normal in $(N_S(P))^g = N_S(P)$. Hence by (i), $\langle P, P^g \rangle = P \times P^g$ and $P \cong P^g$ is isomorphic to a subgroup of $N_S(P)/P$.

LEMMA 9. *Let $G \in I(k)$. Then $H \leqq G$ implies $H \in I(k)$, and H normal in G implies $G/H \in I(k)$.*

PROOF. The first assertion is trivial. Let $G \in I(k)$, H normal in G, and $G/H \notin I(K)$. Let $\{\bar{S}_i\}_{i=1}^{k+1} \subseteq \mathrm{Syl}_2(G/H)$ such that $\bigcap_{i=1}^{k+1} \bar{S}_i \neq H$. For any $1 \leqq i \leqq k+1$, $\bar{S}_i$ is of the form S_iH/H where S_i is a Sylow 2-subgroup of G. Let $H \neq gH \in \bigcap_{i=1}^{k+1} \bar{S}_i$ such that $g \in S_1$. Then there exist $h_i \in H$, for $i = 2, \cdots, k+1$, such that $g \in S_i^{h_i}$. Now $S_1, S_2^{h_2}, \cdots, S_{k+1}^{h_{k+1}}$ are distinct Sylow 2-subgroups because the $\bar{S}_i$ are distinct, hence $1 \neq g \in S_1 \cap S_2^{h_2} \cap \cdots \cap S_{k+1}^{h_{k+1}}$ contradicting $G \in I(k)$.

LEMMA 10. *Assume, in addition to the assumptions of Theorem 1, that the intersection of any two distinct 2-Sylow subgroups of G is of 2-rank $\leqq 1$. Then Theorem 1 holds.*

PROOF. By Lemmas 6 and 9, all we need to prove is that the groups listed in Aschbacher's theorem [1, (1)–(5)] that do not appear in the conclusion of Theorem 1, do not satisfy $I(3)$.

A central product of two copies of $SL_2(5)$ with amalgamated centers, has a non-trivial center contained in all its Sylow 2-subgroups. There are more than three such subgroups and therefore $G \notin I(3)$. The same holds for its extension by an automorphism which permutes the copies of $SL_2(5)$.

In J_1, for any involution t, $C_G(t) = \langle t \rangle \times H$ (for $H \simeq A_5$) contains 5 Sylow 2-subgroups of G intersecting in $\langle t \rangle$, hence $J_1 \notin I(3)$. In a perfect nontrivial central extension of A_7 by a 2-group this 2-group is certainly contained in any 2-Sylow subgroup and again $G \notin I(3)$. In $L_2(q)$, $q = 3, 5 \pmod 8$, $|L_2(q)|_2 = 4$ and there exists a dihedral subgroup L of order $2l$ where $l = \frac{1}{2}(q+1)$ if $q \equiv 3 \pmod 8$ and $l = \frac{1}{2}(q-1)$ if $q \equiv 5 \pmod 8$. L contains $\frac{1}{2}l$ 4-groups intersecting in the central involution of L, so that $\frac{1}{2}l \leqq 3$ whence $q = 3,5,11$ or 13.

$SL_2(q) = L_2(q)$ for even q. For odd q, $|Z(SL_2(q))| = 2$ and we must limit ourselves to q for which $|\mathrm{Syl}_2(SL_2(q))| \leqq 3$. Assume $q > 3$. In that case $L_2(q)$ is simple and $Z \equiv Z(SL_2(q)) = O_2(SL_2(q)) \in C_2(SL_2(q))$. Thus, by the fact that $\sigma(Z) = 3$, and by Lemma 4, $2 = |N_S(Z)/Z| = |S/Z| = \frac{1}{2}|SL_2(q)|_2$. Hence $|SL_2(q)|_2 = 4$, and $|L_2(q)|_2 = 2$ in contradiction to $q > 3$.

PROOF OF THEOREM 1. By Lemma 10 we may assume the existence of two distinct Sylow 2-subgroups S, S' such that $T \equiv S \cap S'$ is of 2-rank > 1. By Lemma 6, $\sigma(T) \geqq 3$, hence by the assumption of the theorem $\sigma(T) = 3$, $T \in \min G$ and by Corollary 7, $\min T = \Sigma(T)$. By Corollary 5 $|N_S(T)/T| = 2$ for every $S \in \Sigma(T)$.

If $|\mathrm{Syl}_2(G)| = 3$ then $T \equiv O_2(G)$ whence $|G/T|_2 = 2$; thus by Burnside, G/T has a normal 2-complement and $O_{2,2',2}(G) = G$. In that case Theorem 1 holds and we may assume that $|\mathrm{Syl}_2(G)| > 3$ and $O_2(G) = 1$.

If for some $S \in \Sigma(T)$ T is not normal in S, then by Lemma 8 (iii)

$$|T| \leqq |N_S(T)/T| = 2$$

whence T is cyclic in contradiction to rank $T > 1$. Hence we may assume T is normal in S for every $S \in \Sigma(T)$, so that $|S : T| = 2$ for every $S \in \Sigma(T)$.

If for every $S \in \Sigma(T)$ and every $S' \notin \Sigma(T)$ we have $S \cap S' = 1$, then by Bender [2], $K \equiv N_G(T)$ is a strongly embedded subgroup of G. By the opening assumption this is no the case, therefore we may assume the existence of $S \in \Sigma(T)$, $S' \notin \Sigma(T)$ such that $S \cap S' \neq 1$. Since $T \in \min G$, $S' \cap T = 1$ and thus

$$|S \cap S'| \leqq \left|\frac{(S \cap S')T}{T}\right| \leqq |S/T| = 2.$$

Let us denote by t the involution of $S \cap S'$. Again since $\sigma(\langle t \rangle) = 3$, $\Sigma(\langle t \rangle) = \min \langle t \rangle$ and so $|N_{\hat{S}}(\langle t \rangle)/\langle t \rangle| = |C_{\hat{S}}(t)/\langle t \rangle| = 2$ for every $\hat{S} \in \Sigma(\langle t \rangle)$. In that case Suzuki [6, Lem. 4] tells us that S is dihedral or semi-dihedral, and T as a maximal subgroup of rank > 1 is dihedral. Now $t \notin Z(\hat{S})$ for every $\hat{S} \in \Sigma(\langle t \rangle)$, since otherwise $|\hat{S}| \leqq |C_{\hat{S}}(t)| = 4$ in contradiction to rank $T > 1$. If G has three conjugate classes of involutions then S is dihedral and by [4, (7.7.3)] G has a normal 2-complement, whence the theorem is proved. Hence we may assume that this is not the case and that G has exactly two conjugate classes of involutions. Since $T \in \min G$, no involution of T can be conjugate to t, which in turn is conjugate to every involution of $S \setminus T$ because S is dihedral or semidihedral. Hence every involution of T is central, whence it is central in one of the three elements of $\Sigma(T)$. Each one of these has only one central involution, whence T has at most three involutions. Concluding, $T \simeq E_4$ and $S \simeq D_8$.

$\bar{G} = G/O(G)$ satisfies $I(3)$ by Lemma 9. The homomorphism $\phi: G \to \bar{G}$ sends Sylow 2-subgroups to isomorphic images and central involutions to central involutions. Therefore $\bar{T}$ contains three central involutions, whence $\bar{T}$ is contained in $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ which are distinct Sylow 2-subgroups of $\bar{G}$, dihedral of order 8.

Applying the same argument to $\bar{G}$ we see that either $\bar{G}$ has only three Sylow 2-subgroups, or $\bar{G}$ has a strongly embedded subgroup, or $\bar{G}$ contains some $\bar{S} \in \Sigma(\bar{T})$, $\bar{S}' \notin \Sigma(\bar{T})$ such that $\bar{S} \cap \bar{S}' = \langle t \rangle$ of order 2. In the first case $O_{2,2',2}(\bar{G}) = \bar{G}$, $\bar{G}$ is solvable, and by [4 (16.3)] either $\bar{G}$ is a 2-group or $\bar{G} \cong P\Gamma L_2(3) \equiv PGL_2(3)$ and we are through. The second case cannot occur since a Sylow 2-subgroup of $\bar{G}$ is dihedral of order 8. Therefore we are left with the third case, where $\bar{G}$ must have exactly two conjugate classes of involutions unless $\bar{G}$ is solvable. Hence by [4, (16.3)], H_1 is normal in $\bar{G} \leqq H_2$ where $H_1 \simeq L_2(q)$, $H_2 \simeq P\Gamma L_2(q)$ such that $|\bar{G}: H_1|$ is even (for otherwise there would be only one conjugate class of involutions). Therefore $|H_1|_2 = 4$ and $q \equiv \pm 3(8)$. Moreover, $H_1 \cap \bar{S}_1 \in \mathrm{Syl}_2(H_1)$ so that the involutions of $H_1 \cap \bar{S}_1$ are conjugate and $\bar{S}_1 \cap H_1 = \bar{T}$. But $T \cap T^g \neq 1$ implies $T = T^g$ for all $g \in G$, hence for all $\bar{g} \in H_1$. It follows that any two Sylow 2-subgroups of H_1 must intersect trivially whence, by an argument of Lemma 10, $q \leqq 5$. Concluding, $H_1 \simeq PSL_2(5)$ and

$$H_2 \simeq P\Gamma L_2(5) \simeq PGL_2(5)$$

implies $|H_2: H_1| = 2$, thus $\bar{G} \simeq PGL_2(5)$.

LEMMA 11. *Let G be a finite simple group and let $T \in \min G$ be a 2-subgroup of G with $\sigma(T) = 5$ and $\min_3 T \neq \varnothing$. Then a Sylow 2-subgroup of G is either dihedral or semidihedral.*

PROOF. Assume $\Sigma(T) = \{S_i\}_{i=1}^5$ so that $T = \bigcap_{i=1}^5 S_i$. Since $\min_3 T \neq \varnothing$ we may assume that $T_1 \equiv \bigcap_{i=1}^3 S_i > T$. If $\min_3 T = \{T_1\}$, then $\{S_1^g, S_2^g, S_3^g\} = \{S_1, S_2, S_3\}$ for all $g \in N_G(T)$. In addition it follows that $S_4 \in \min T$, whence by Lemma 4 $N_{S_4}(T)/T$ moves S_5 to say S_1, a contradiction.

Therefore we may assume that $\min_3 T \supset \{T_1\}$, so there exists $\min_3 T \ni T_2$

$= S_i \cap S_j \cap S_k$. Since $\sigma(T_1) = \sigma(T_2) = 3$, $\min T_1 = \Sigma(T_1)$ and $\min T_2 = \Sigma(T_2)$ so that $|\{1,2,3\} \cap \{i,j,k\}| \leqq 1$. Hence we may assume that $T_2 = S_1 \cap S_4 \cap S_5$. The same argument applied again yields $\min_3 T = \{T_1, T_2\}$.

Now $T_1 \in \min T$ implies that $N_{T_1}(T)/T$ acts faithfully on $\Sigma(T)$ and fixing $\{S_1, S_2, S_3\}$ it must be of order 2. But the structure of $\min_3 T$ forces $N_G(T)$ to stabilize S_1, so that $N_{S_2}(T)/T = N_{S_1 \cap S_2}(T)/T = N_{T_1}(T)/T$, whence $|N_{S_2}(T): T| = 2$.

Since $\min_3 T \neq \emptyset$, it follows that T is not normal in S_2 and by Lemma 8(iii), $|\langle t \rangle| = |T| = 2$ and $|C_S(t)| = 4$. Hence we are through by Suzuki [6, Lem. 4].

LEMMA 12. *Let G be a finite simple group, with a dihedral or semi-dihedral. Sylow 2-subgroup, such that $G \in I(5)$. Then G is isomorphic to one of the following*:

(i) $L_2(q), q = 5,7,9,11,13,19$,

(ii) $L_3(3)$, or

(iii) M_{11}.

PROOF. If $|S| = 4$ then $G \simeq L_2(q)$ for $q \equiv 3,5 \pmod 8$; by an argument analogous to that of Lemma 10 we conclude that $q = 3,5,11,13,17$ or 19. Hence we may assume that $|S| \geqq 8$.

By [4, (7.7.3)] for the dihedral case and [4, (7), Ex. 7] for the semidihedral case, the simplicity of G implies that there is only one conjugate class of involutions. Hence for any involution z, $|C_G(z)|_2 = |G|_2$. For any involution $t \in C_G(z) \setminus \langle z \rangle$, $z \in C_G(t)$ so that $z \in T$ for some $T \in \mathrm{Syl}_2(C_G(t)) \subseteq \mathrm{Syl}_2(G)$. As for any $S \in \mathrm{Syl}_2(G)$, $|\Omega(Z(S))^{\#}| = 1$, $\sigma(\langle z \rangle) \geqq |\mathrm{Syl}_2(C_G(z))| +$ the number of involutions in $C_G(z) \setminus \langle z \rangle$. But the number of involutions in $C_G(z) \setminus \langle z \rangle$ is greater or equal to four, and $|\mathrm{Syl}_2(C_G(z))| \geqq 1$, hence $\sigma(\langle z \rangle) \leqq 5$ implies that $|\mathrm{Syl}_2(C_G(z)| = 1$ and that $|S| = 8$ if S is dihedral and $|S| = 16$ if S is semidihedral.

If S is dihedral and $G \simeq A_7$, one calculates $C_G(z)$ for $z = (12)(34)$ to show that $|\mathrm{Syl}_2(C_G(z))| = 3$.

If S is dihedral and $G \simeq L_2(q)$, $q \equiv \pm 1 \pmod 8$, G contains a dihedral subgroup K of order $2l$ where $l = (q-1)/2$ for $q \equiv 1 \pmod 8$ and $l = (q+1)/2$ for $q \equiv -1 \pmod 8$. Moreover $|K|_2 = |G|_2$ and if we write $l = 2^k \cdot m$, for m odd, then $|\mathrm{Syl}_2(K)| = m$ and $|K|_2 = 2^{k+1}$. Hence if z is the central involution of K, $K \subseteq C_G(z)$ implies $m = 1$, $2^{k+1} = 8$, whence $q = 7$ or $q = 9$. By [4, (16.3)] we are through in case S is dihedral.

If S is semidihedral then $|\mathrm{Syl}_2(C_G(z))| = 1$ together with the fact that G has only one conjugate class of involutions imply that the centralizer of any involution is 2-closed, hence by Wong [4, p. 288], $G \simeq L_3(3)$ or $G \simeq M_{11}$.

LEMMA 13. *Let G be a finite group with $O_2(G) = 1$ and $O^2(G) = G$. Let T be a 2-subgroup of G satisfying $T \in \min G$ and $|N_S(T): T| = 2$ for some $S \in \Sigma(T)$. Then either G contains a strongly embedded subgroup or a Sylow 2-subgroup of G is dihedral or semidihedral.*

PROOF. If T is not normal in S then by Lemma 8(iii) $|T| = 2$; thus by Suzuki [6, Lem. 4] S is dihedral or semidihedral. Hence we may assume that T is normal in S, so that $|T| = |G|_2/2$ and T is of index two in every $S \in \Sigma(T)$.

If for every $S \in \Sigma(T)$ and every $S' \notin \Sigma(T)$ we have $S \cap S' = 1$, we are following Bender [2]. Hence we may assume that for some $S \in \Sigma(T)$, $S' \notin \Sigma(T)$, $S \cap S' \neq 1$. But $T \in \min G$ implies that $S' \cap T = 1$ so that $S \cap S' = \langle t \rangle$ where t is an involution. By the N-group [7, (5.38)] paper there exists some $g \in G$ for which $t^g \in T$, whence $T \geqq S \cap S^g \cap S'^g > 1$. Now $T \in \min G$ implies that $T = S \cap S^g \cap S'^g$ whence $|T| = 2$ and S is a 4-group, hence dihedral.

LEMMA 14. *Let G be a finite simple group satisfying $I(5)$. If G has a strongly closed abelian 2-group then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$, $U_3(q)$, for q even, $L_2(q)$, for $q = 5,11,13,19$ or J_1.*

PROOF. Following Goldschmidt [3] we need check only two things, namely, that the above-mentioned values of $q \equiv 3, 5 \pmod 8$ are the only ones for which $L_2(q) \in I(5)$, and that the only group of Janko-Ree type [3, Case e] which occurs here is J_1. The first check is a word-by-word repetition of the analogous part of Lemma 10. In Case e, G has a subgroup G_1 of odd index such that $G_1 = \langle t \rangle \times K$ where t is an involution and $K \simeq L_2(q)$, $q \equiv 3, 5 \pmod 8$, $q > 3$ with $|\mathrm{Syl}_2(K)| \leqq 5$. Hence $G \simeq J_1$.

PROOF OF THEOREM 3. If G has a strongly embedded subgroup we are through by [2]. By Corollary 5 and by Lemmas 11, 12 and 13 we may assume that for every $T \in \min G$, $\sigma(T) = 5$, $\min T = \Sigma(T)$ and $|N_S(T): T| = 4$ for every $S \in \Sigma(T)$. In such a situation $C_2(G) = \{1\} \cup \min G \cup \mathrm{Syl}_2(G)$.

If for some $S \in \mathrm{Syl}_2(G)$ and for $z \in \Omega(Z(S))$ we have $\Sigma(\langle z \rangle) = \{S\}$, then we are through by Herzog [5]; thus we may assume that for every $S \in \mathrm{Syl}_2(G)$ and every $z \in \Omega(Z(S))$ there exists some $S' \in \mathrm{Syl}_2(G)$ such that $z \in T \equiv S \cap S' \in \min G$. Since

$T \cap Z(S) \neq 1$, T is normal in S by Lemma 8(ii), and we conclude that for every $S \in \mathrm{Syl}_2(G)$ there exists $T \in \min G$ such that T is normal in S.

Let us fix T and S. First assume that there exists some $T' \neq T$ such that $T' \sim_G T$ and $T' \leqq S$. It follows that $T' \in \min G$, T' is normal in S (because $|T| = |T'|$, $|N_S(T)/T| = |N_S(T')/T'| = 4$), and $T \cap T' = 1$. Hence

$$|N_S(T)/T| = |N_S(T')/T'| = 4$$

implies that $S \simeq E_{16}$ or $|S: E_4| = 2$. In the latter case S contains at least two central involutions whence $S \simeq E_8$ or $S \simeq E_4 \times E_2$. In any case, S is abelian and we are through by Lemma 14.

Otherwise define $V \equiv \langle z \in \Omega(Z(S')) \cap T \mid S' \in \Sigma(T) \rangle$. Since $T \leqq S'$ for every $S' \in \Sigma(T)$, $1 < V \leqq Z(T)$ so that V is an abelian 2-subgroup of G. We claim that V is strongly closed in S with respect to G. If we show this we are through by quoting Lemma 14 again. Indeed, let $z \in V$ and let $z^g \in S$. Now $T^g \cap S \ni z^g \neq 1$ and $T^g \in \min G$, therefore $S \geqq T^g$ whence $T = T^g$. Hence $z^g \in T$ and $z^g \in \Omega(Z(S^g))$ where $S^g \in \Sigma(T)$, thus $z^g \in V$ and V is strongly closed in S with respect to G.

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